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Simplicial powers of graphs

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ABSTRACT

In a graph, a vertex is simplicial if its neighborhood is a clique. For an integer $k \geq 1$, a graph $G = (V_G, E_G)$ is the *k-simplicial power* of a graph $H = (V_H, E_H)$ (H a *root graph* of G) if V_G is the set of all simplicial vertices of H, and for all distinct vertices x and y in V_G, $xy \in E_G$ if and only if the distance in H between x and y is at most k. This concept generalizes k-leaf powers introduced by Nishimura, Ragde and Thilikos which were motivated by the search for underlying phylogenetic trees; k-leaf powers are the k-simplicial powers of trees. Recently, a lot of work has been done on k-leaf powers and their roots as well as on their variants phylogenetic roots and Steiner roots. For $k \leq 5$, k-leaf powers can be recognized in linear time, and for $k \leq 4$, structural characterizations are known. For $k \geq 6$, the recognition and characterization problems of k-leaf powers are still open. Since trees and block graphs (i.e., connected graphs whose blocks are cliques) have very similar metric properties, it is natural to study k-simplicial powers of block graphs. We show that leaf powers of trees and simplicial powers of block graphs are closely related, and we study simplicial powers of other graph classes containing all trees such as ptolemaic graphs and strongly chordal graphs.

1. Introduction

Motivated by background from phylogenetic trees [2,15,30], Nishimura, Ragde and Thilikos [28] introduced the following notion: For an integer $k \geq 2$, a finite undirected graph $G = (V_G, E_G)$ is a *k-leaf power* if there is a tree T with V_G as its set of leaves such that for all distinct $x, y \in V_G$, $xy \in E_G$ if and only if the distance between x and y in T is at most k. Then T is called a *k-leaf root of G*. In general, a *leaf power* is a k-leaf power for some $k \geq 2$.

Obviously, a graph is a 2-leaf power if and only if it is a disjoint union of cliques. In [28], a (very complicated) $\mathcal{O}(n^3)$ time algorithm for recognizing 3-leaf powers and 4-leaf powers, respectively, and constructing 3-leaf roots and 4-leaf roots, respectively, if they exist, was described. Recently, Chang and Ko [14] gave a linear time recognition algorithm for 5-leaf powers. Despite considerable effort, for $k \geq 6$, no non-trivial characterization and no efficient recognition of k-leaf powers is known. See [5–7,10,29] for more information on leaf powers and in particular, for new characterizations of 3- and 4-leaf powers as well as of distance-hereditary 5-leaf powers.

It is known that for every $k \geq 2$, k-leaf powers are strongly chordal [6]. In [3], Bibelnieks and Dearing introduced and studied so-called NeST graphs (i.e., neighborhood subtree tolerance graphs); for constant tolerances these are exactly the induced subgraphs of powers of trees, i.e., equivalently, k-leaf powers (see [8,21]). In [3], an example of a graph is given which is strongly chordal but no fixed tolerance NeST graph (i.e., no k-leaf power for any k), and in [21] this is slightly generalized; [21] mentions the problem of characterizing fixed tolerance NeST graphs (i.e., k-leaf powers for some k).

Our aim is to generalize the notion of leaf powers of trees to the very natural notion of simplicial powers of graphs which is also of independent interest; a vertex is *simplicial* if its neighborhood is a clique:

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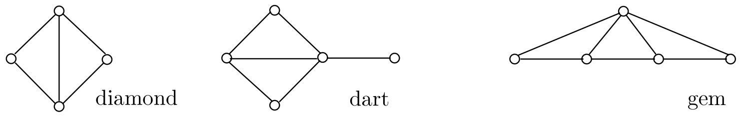

Fig. 1. Three small graphs.

A graph $G = (V_G, E_G)$ is the *k-simplicial power of another graph* $H = (V_H, E_H)$ if $V_G \subseteq V_H$ is the set of all simplicial vertices of H, and for all distinct vertices $x, y \in V_G$, $xy \in E_G$ if and only if the distance in H between x and y is at most k.

Since trees and block graphs have very similar metric properties (cf. [12,23]), it is natural to study k-simplicial powers of block graphs. In particular, the motivation of this paper comes from Theorem 7 in Section 4 which claims that for $k \geq 2$, a graph is the k-leaf power of a tree if and only if it is the $(k-1)$-simplicial power of a claw-free block graph. Thus, our focus is on simplicial powers of block graphs but we also consider other graph classes containing all trees such as ptolemaic graphs and strongly chordal graphs.

2. Preliminaries

Throughout this paper, let $G = (V_G, E_G)$ denote a finite undirected graph without loops and multiple edges, with vertex set V_G and edge set E_G. For a vertex $v \in V_G$, let $N_G(v) = \{u \mid uv \in E_G\}$ denote the *neighborhood* of v in G, and let $N_G[v] = \{v\} \cup N_G(v)$ denote the *closed neighborhood* of v in G. The *degree* $\deg_G(v)$ of a vertex v is the number of its neighbors, i.e., $\deg_G(v) = |N_G(v)|$. The *complement graph of* G is denoted by $\overline{G}$. A *clique* (*stable set*) is a set of mutually (non-)adjacent vertices. A vertex v of G is *simplicial* in G if $N_G(v)$ is a clique; vertices of degree one are simplicial and they are called *leaves*. A vertex is *universal* if it is adjacent to all other vertices. Two vertices x and y are *true twins* if $N_G[x] = N_G[y]$.

A *cut vertex* is a vertex whose removal increases the number of connected components. A connected graph is 2-*connected* if it has no cut vertex. As usual, the maximal induced 2-connected subgraphs of G are the *blocks* (or 2-*connected components*) of G. A block of G which contains at most one cut vertex is an *endblock*.

For $U \subseteq V_G$, let $G[U]$ denote the subgraph of G induced by U. For a set $\mathcal{F}$ of graphs, a graph is $\mathcal{F}$*-free* if none of its induced subgraphs is (isomorphic to a graph) in $\mathcal{F}$.

Replacing a vertex v in a graph G by a graph H (or *substituting* H for v) results in the graph obtained from $G[V_G \setminus \{v\}] \cup H$ by adding all edges between vertices in $N_G(v)$ and vertices in H.

For an integer $\ell \geq 1$, let P_ℓ denote the chordless path with ℓ vertices and $\ell - 1$ edges, and for $\ell \geq 3$, let C_ℓ denote the chordless cycle with ℓ vertices and ℓ edges. A complete bipartite graph with r vertices in one color class and s vertices in the other color class is denoted by $K_{r,s}$; the $K_{1,3}$ is also called the *claw*. For $\ell \geq 3$, let S_ℓ denote the (*complete*) *sun* with 2ℓ vertices $u_1, \ldots, u_\ell$ and $w_1, \ldots, w_\ell$ such that $\{u_1, \ldots, u_\ell\}$ is a clique, $\{w_1, \ldots, w_\ell\}$ is a stable set and for $i \in \{1, \ldots, \ell\}$, w_i is adjacent to exactly u_i and u_{i+1} (index arithmetic modulo ℓ). A graph is *sun-free* if it contains no induced S_ℓ for all $\ell \geq 3$.

A graph is *chordal* if it contains no induced C_ℓ with $\ell \geq 4$. A graph is *strongly chordal* if it is chordal and sun-free. It is known that leaf powers are strongly chordal (cf. [6], Proposition 3). A graph is a *split graph* if its vertex set can be partitioned into a clique and a stable set. Clearly, G is a split graph if and only if G and its complement graph $\overline{G}$ are chordal. A graph is *ptolemaic* if it is chordal and gem-free (see Fig. 1 for the gem).

Let $\mathcal{S}$ be a family of nonempty sets. The *intersection graph* of $\mathcal{S}$ is obtained by representing each set in $\mathcal{S}$ by a vertex and connecting two vertices by an edge if their corresponding sets intersect.

A connected graph is a *block graph* if each of its blocks is a clique. Clearly, block graphs are ptolemaic but not vice-versa. As block graphs will play a crucial role in this paper, we give in Theorem 1 some well-known characterizations of them; the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 1 is Theorem 3.5 in [22], and the equivalence (i) $\Leftrightarrow$ (iii) can be easily seen, e.g., by [10, Observation 3].

Theorem 1. *For every connected graph G, the following statements are equivalent:*

(i) *G is a block graph;*
(ii) *G is the intersection graph of the blocks of some graph;*
(iii) *G is chordal and diamond-free.*

Let $d_G(x, y)$ denote the distance in G between x and y (i.e., the minimum number of edges of a path in G connecting x and y). Let $G^k = (V_G, E_G^k)$ with $xy \in E_G^k$ if and only if $1 \leq d_G(x, y) \leq k$ denote the k-th *power of* G.

A set system $\mathcal{E}$ has the *Helly property* if for every pairwise intersecting subsystem $\mathcal{E}' \subseteq \mathcal{E}$, the total intersection $\bigcap \mathcal{E}'$ is nonempty. A graph G is *clique-Helly* if the set $\mathcal{C}(G)$ of its maximal cliques has the Helly property.

Let $G = (V_G, E_G)$ be a graph. Its *line graph* $L(G)$ is the intersection graph of E_G, i.e., $L(G)$ has E_G as its vertex set, and two distinct edges e, e' are adjacent in $L(G)$ if and only if $e \cap e' \neq \emptyset$. The *clique graph* $K(G)$ of G is the intersection graph of $\mathcal{C}(G)$, i.e., $K(G)$ has $\mathcal{C}(G)$ as its vertex set, and two distinct maximal cliques Q, Q' are adjacent if and only if $Q \cap Q' \neq \emptyset$.

The bipartite *vertex-clique incidence graph* $B_{\mathcal{C}}(G)$ of G has V_G and $\mathcal{C}(G)$ as its color classes, and vQ is an edge in $B_{\mathcal{C}}(G)$ if and only if $v \in Q$. Let $split(B_{\mathcal{C}}(G))$ denote the split graph resulting from $B_{\mathcal{C}}(G)$ by completing $\mathcal{C}(G)$ to a clique. Obviously, $split(B_{\mathcal{C}}(G))$ is a split graph with V_G as its set of simplicial vertices.

We will make use of the following well-known facts (see [9] for a survey); bipartite graphs without induced C_ℓ for any $\ell \geq 6$ are called *chordal bipartite*.

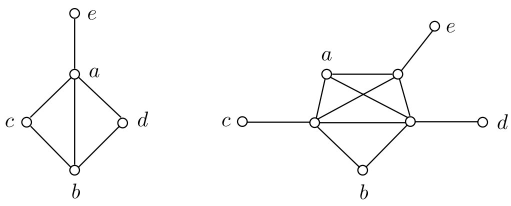

Fig. 2. A 2-simplicial power and one of its 2-simplicial roots.

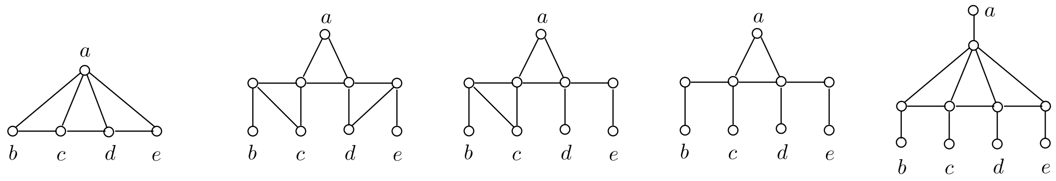

Fig. 3. A 3-simplicial power and some of its 3-simplicial roots.

Theorem 2.

(i) *Powers of strongly chordal graphs are strongly chordal* [17,26,27].
(ii) *Strongly chordal graphs are clique-Helly* (see, e.g., [4]).
(iii) *G is strongly chordal if and only if $B_C(G)$ is chordal bipartite if and only if split $(B_C(G))$ is strongly chordal* ([19], see also [9]).

In particular, ptolemaic graphs are clique-Helly and all powers of ptolemaic graphs are strongly chordal. We will also make use of the following property of ptolemaic graphs.

Theorem 3 (*[1]*). *The clique graph of a ptolemaic graph is ptolemaic.*

3. Simplicial powers and general results

The key notion of this paper is the following natural generalization of leaf powers.

Definition 1. Let $k \geq 1$ be an integer. A graph $G = (V_G, E_G)$ is the *k-simplicial power* of a graph $H = (V_H, E_H)$ if $V_G \subseteq V_H$ is the set of all simplicial vertices in H and for all $x, y \in V_G, x \neq y, xy \in E_G$ if and only if $d_H(x, y) \leq k$. Then such a graph H is a *k-simplicial root* of G.

If G is the k-simplicial power of H and if, in addition, V_G consists of exactly the degree-1 vertices, i.e., leaves of H, then we also say that G is the *k-leaf power of H*.

Remark 1. If $G = (V_G, E_G)$ is the k-simplicial power of a graph H, then G is an induced subgraph of the usual k-th power H^k of H; namely $G = H^k[V_G]$.

Remark 2. Note that in a triangle-free graph, any simplicial vertex has degree 1, i.e., simplicial vertices coincide with leaves. Thus, in the sense of [28], leaf powers are exactly the simplicial powers of trees. We remark that in Definition 1, instead of requiring that V_G is the set of all simplicial vertices in H, one may only require that V_G is *some* subset of V_H. In particular, if we do so, if $V_G = V_H$, then $G = H^k$.

Obviously, a graph is the 1-simplicial power of some graph if and only if it is a disjoint union of cliques, i.e., it does not contain an induced path P_3 on three vertices.

Figs. 2 and 3 give some examples of 2- and 3-simplicial powers and some of their roots.

As we will see in Proposition 1 and Theorem 4, every graph is the 2-simplicial power of some split graph and the simplicial power of some graph. Thus, the notion is only interesting for some very restricted types of root graphs.

We often use the following constructions for simplicial roots.

Lemma 1. *Let $H = (V_H, E_H)$ be a graph and let $H^* = (V_{H^*}, E_{H^*})$ with $V_{H^*} = V_H \cup \{v^* \mid v$ is a simplicial vertex of $H\}$ and $E_{H^*} = E_H \cup \{vv^* \mid v \in V_H\}$. If H is a k-simplicial root of a graph G then H^* is a $(k+2)$-leaf root of the graph G.*

Proof. Clearly, the vertices v^* of H^*, $v \in V_H$, are exactly the simplicial vertices (all are leaves) of the graph H^*. Moreover, for all $u, v \in V_H$, $d_{H^*}(u^*, v^*) = d_H(u, v) + 2$, hence $uv \in E_G$ if and only if $d_{H^*}(u, v) \leq k+2$. Thus, G is the $(k+2)$-leaf power of the graph H^*. $\square$

Note that if H belongs to a particular graph class such as trees, block graphs, bipartite graphs or chordal graphs, then the graph H^* in Lemma 1 belongs to the same graph class. Also, if H is a split graph, then H^* is chordal.

Proposition 1. *Every graph is*

(i) *the* 2*-simplicial power of a* split *graph, and*
(ii) *the* 4*-leaf power of a* bipartite *graph.*

Proof. Let $G = (V_G, E_G)$ be an arbitrary graph.

(i): Let $H = split(B_{\mathcal{C}}(G))$. Then V_G is the set of simplicial vertices of the split graph H and, by definition, for all vertices $x, y \in V_G$ we have: $xy \in E_G$ if and only if $x, y \in C$ for some maximal clique C of G if and only if xC and yC are edges in H if and only if $d_H(x, y) \leq 2$. Thus, G is the 2-simplicial power of the split graph H.

(ii): Let $H = B_{\mathcal{C}}(G)$. By definition, for all vertices $x, y \in V_G$, we have: $xy \in E_G$ if and only if $x, y \in C$ for some maximal clique C of G if and only if xC and yC are edges in H if and only if $d_H(x, y) \leq 2$.

Let H' be the graph obtained from H by adding a new vertex v' for each vertex $v \in V_G$ and adding the new edge $v'v$. Clearly, H' is bipartite and the set of the new vertices v' is exactly the set of simplicial vertices (all are leaves) of H'. Moreover, for all $u, v \in V_G$, $d_{H'}(u', v') = d_H(u, v) + 2$, hence $uv \in E_G$ if and only if $d_{H'}(u', v') = 4$. Thus, G is the 4-leaf power of the bipartite graph H'. □

In view of Proposition 1(i) it is interesting to note that 3-simplicial powers of split graphs are cliques. From Proposition 1 and Lemma 1, we obtain:

Corollary 1. *For all even $k \geq 2$ and all even $\ell \geq 4$, every graph is*

(i) *the k-simplicial power of a chordal graph, and*
(ii) *the ℓ-leaf power of a bipartite graph.*

Theorem 4. *For each $k \geq 2$, every graph is the k-simplicial power of some graph.*

Proof. The case of even k has been shown in Corollary 1. For odd k, let $G = (V_G, E_G)$ be an arbitrary connected graph. Let H be the graph obtained from G by taking a new vertex v' for each vertex $v \in V_G$ and adding the new edge $v'v$. Clearly, the new vertices v' are exactly the simplicial vertices of the graph H. Moreover, for all $u, v \in V_G$, $d_H(u', v') = d_G(u, v) + 2$, hence $uv \in E_G$ if and only if $d_H(u', v') = 3$. Thus, G is the 3-leaf power of the graph H. Hence, by Lemma 1, Theorem 4 is proved. □

In the proof of Proposition 1(i) a split root graph of a given graph G is constructed which might be exponentially larger than G. This suggests the following problem.

2-SIMPLICIAL SPLIT GRAPH ROOT

Instance: A graph $G = (V_G, E_G)$ and an integer ℓ.
Question: Does there exist a split graph $H = (V_H, E_H)$ with $|V_H| \leq \ell$ such that G is the 2-simplicial power of H?

Theorem 5. 2-SIMPLICIAL SPLIT GRAPH ROOT *is NP-complete.*

Proof. 2-SIMPLICIAL SPLIT GRAPH ROOT is clearly in NP. We reduce the following NP-complete problem [20, GT59] to our problem:

INTERSECTION GRAPH BASIS

Instance: A graph $G = (V_G, E_G)$ and an integer ℓ.
Question: Does there exist a set S with $|S| \leq \ell$ and subsets $S_v \subseteq S$, $v \in V_G$, such that $uv \in E_G$ if and only if $S_u \cap S_v \neq \emptyset$?

For given graph $G = (V_G, E_G)$ and integer ℓ, we claim that G has an intersection basis S with $|S| \leq \ell$ if and only if G has a 2-simplicial split graph root $H = (V_H, E_H)$ with $|V_H| \leq |V_G| + \ell$.

First, let G have an intersection basis S with $|S| \leq \ell$. We may assume that every $s \in S$ belongs to at least two subsets $S_v, S_{v'}$ for some $v \neq v'$ (if not, set $S := S - s$). Let $H = (V_H, E_H)$ be the split graph with $V_H = V_G \cup S$ and $E_H = \{ss' : s, s' \in S, s \neq s'\} \cup \{vs : v \in V_G, s \in S, s \in S_v\}$. Then V_G consists of exactly the simplicial vertices in H, and $uv \in E_G \Leftrightarrow S_u \cap S_v \neq \emptyset \Leftrightarrow N_H(u) \cap N_H(v) \neq \emptyset \Leftrightarrow d_H(u, v) = 2$. That is, G is the 2-simplicial power of H. Moreover, $|V_H| = |V_G| + |S| \leq |V_G| + \ell$.

Next, let G have a 2-simplicial split graph root $H = (V_H, E_H)$ with $|V_H| \leq |V_G| + \ell$. Consider a partition $V_H = Q \cup S$ with a stable set Q and a clique S. We may assume that no vertex in S is a simplicial vertex of H (if $s \in S$ is simplicial, s can have at most one neighbor in Q, say q if any. Then $H' = (Q + s \cup S - s, E_H - sq)$ is also a 2-simplicial split graph root of G with the same order). Thus, $V_G = Q$, hence $|S| \leq \ell$. Moreover, $uv \in E_G \Leftrightarrow d_H(u, v) = 2 \Leftrightarrow N_H(u) \cap N_H(v) \neq \emptyset$. Setting $S_v = N_H(v) \subseteq S$, $v \in V_G$, S is an intersection basis of G with at most ℓ elements. □

Also, in the proof of Proposition 1(ii) a bipartite root graph of a given graph G is constructed which might be exponentially larger than G. This suggests the following problem.

4-LEAF BIPARTITE GRAPH ROOT

Instance: A graph $G = (V_G, E_G)$ and an integer ℓ.
Question: Does there exist a bipartite graph $H = (V_H, E_H)$ with $|V_H| \leq \ell$ such that G is the 4-leaf power of H?

The computational complexity of this problem is not determined yet. Although we strongly believe that 4-LEAF BIPARTITE GRAPH ROOT is NP-complete, we are unable to find a proof. Probably, a non-straightforward modification of the reduction in the proof of Theorem 5 could work.

4. Simplicial powers versus leaf powers

In this section we describe a close relationship between leaf powers (of trees) and simplicial powers of block graphs. Theorem 7 characterizes leaf powers in terms of simplicial powers of claw-free block graphs; this was the main motivation for this paper. As a preparing step, we need:

Theorem 6 ([22], Theorem 8.5). *A graph is the line graph of a tree if and only if it is a claw-free block graph.*

The following observation is straightforward.

Observation 1. *Let T be a tree.*
(i) *An edge xy of T is a simplicial vertex in its line graph $L(T)$ if and only if x or y is a leaf in T.*
(ii) *Let $e_x = xx'$, $e_y = yy'$ be two edges of T where x' and y' are leaves. Then $d_{L(T)}(e_x, e_y) = d_T(x', y') - 1$.*

Theorem 7. *For $k \geq 2$, a graph is the k-leaf power of a tree if and only if it is the $(k-1)$-simplicial power of a claw-free block graph.*

Proof. *Necessity*. Suppose that $G = (V_G, E_G)$ is the k-leaf power of a tree T. To avoid triviality, we assume that T has more than one edge. By Theorem 6 and Observation 1(i), the line graph $L(T)$ is a claw-free block graph whose simplicial vertices are in one-to-one correspondence to the end edges of T (as T has more than one edge). For leaf x in T, let e_x denote the edge in T containing x. Then by Observation 1(ii), $xy \in E_G \Leftrightarrow d_T(x, y) \leq k \Leftrightarrow d_{L(T)}(e_x, e_y) \leq k - 1$ and e_x, e_y are simplicial in $L(T)$.
Sufficiency. Suppose that $G = (V_G, E_G)$ is the $(k-1)$-simplicial power of a claw-free block graph H. By Theorem 6, $H = L(T)$ for a tree T, and we may assume that T has more than one edge. Then the leaves v of T uniquely correspond to the edges e_v of T incident to v. By Observation 1(i), these edges e_v of T are exactly the simplicial vertices of $L(T)$. Moreover, for all distinct leaves v, v' of T, we have $d_T(v, v') = d_{L(T)}(e_v, e_{v'}) + 1$. Hence G is the k-leaf power of the tree T. □

Corollary 2. *The class of k-simplicial powers of block graphs contains all t-leaf powers for $t \leq k + 1$.*

Proof. By Theorem 7, $(k+1)$-leaf powers are k-simplicial powers of block graphs. Furthermore, k-leaf powers are particularly k-simplicial powers of block graphs. Then, Corollary 2 follows by observing that t-leaf powers are $(t+2)$-leaf powers; cf. Lemma 1. □

In view of Theorem 7 and Corollary 2, the larger class of simplicial powers of (not necessarily claw-free) block graphs is an interesting class of graphs and is of independent interest. In the rest of this section we collect some fundamental properties of simplicial powers of block graphs; these are also very related to basic properties of leaf powers given in [10].

Proposition 2.
(i) *An induced subgraph of a k-simplicial power of a block graph is again a k-simplicial power of a block graph;*
(ii) *A graph is a k-simplicial power of a block graph if and only if each of its connected components is a k-simplicial power of a block graph;*
(iii) *k-simplicial powers of block graphs are strongly chordal.*

Proof. The proofs for (i) and (ii) are similar to the case of leaf powers given in [10], hence we omit them. To (iii): This is because of Remark 1 and Theorem 2 (i). □

It is known (see [10]) that leaf powers of trees are exactly those graphs obtainable from an induced subgraph of a tree power by replacing vertices by cliques. A similar statement is true for simplicial powers of block graphs; it is based on the following notion.

Definition 2. A graph is a *basic k-simplicial power of a block graph* if it admits a k-simplicial block graph root in which each block contains at most one simplicial vertex. Such a k-simplicial block graph root is called *basic*.

Examples of basic k-simplicial powers of block graphs include k-leaf powers (any k-leaf tree root T of a k-leaf power G is a basic k-simplicial block graph root of G). Obviously,

> A graph is a k-simplicial power of a block graph if and only if it is obtained from a basic k-simplicial power of a block graph by replacing vertices by cliques. In particular, any k-simplicial power of a block graph without true twins is basic.

We also note that Proposition 2 accordingly remains true for basic k-simplicial powers of block graphs.

Theorem 8. *Let $k \geq 2$ be an integer. A graph is a basic k-simplicial power of a block graph if and only if it is an induced subgraph of the $(k-1)$-th power of a block graph.*

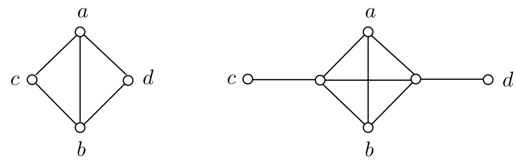

Fig. 4. The diamond and its unique 2-simplicial block graph root.

Proof. *Necessity*. Let $R = (V_R, E_R)$ be a basic k-simplicial block graph root of $G = (V_G, E_G)$. Assume first that every block of R contains exactly one simplicial vertex. Thus, letting B_v be the block of R containing $v \in V_G$, the correspondence $v \mapsto B_v$ is a bijection. Consider the intersection graph H of the blocks of R. By Theorem 1(ii), H is a block graph. Moreover, for all distinct vertices u and v in G,

$$uv \in E_G \Leftrightarrow 1 < d_R(u, v) \leq k \Leftrightarrow 1 \leq d_H(B_u, B_v) \leq k - 1 \Leftrightarrow B_uB_v \in E(H^{k-1}),$$

where the middle equivalence holds as R is a block graph and by definition of H. That is, G is the $(k-1)$-th power of a block graph.

In the case where not every block of R contains a simplicial vertex, consider the block graph R' obtained from R by taking a new vertex b for each block B of R that contains no simplicial vertex and adding all edges between b and all vertices in B. Then R' is a block graph and each of its blocks contains exactly one simplicial vertex. By the previous case, the (basic) k-simplicial power G' of R' is the $(k-1)$-th power of a block graph, and we are done by noting that G is obtained from G' by removing the new vertices.

Sufficiency. Since an induced subgraph of a basic k-simplicial power of a block graph is again a basic k-simplicial power of a block graph, we have only to show that $(k-1)$-th powers of block graphs are basic k-simplicial powers of block graphs. Let $G = (V_G, E_G)$ be the $(k-1)$-th power of a block graph $H = (V_G, E_H)$, and let $\mathcal{B}$ denote the set of all blocks of H. Consider the intersection graph R of $V_G \cup \mathcal{B}$. Since by Theorem 1, the intersection graph of $\mathcal{B}$ is a block graph, R is a block graph in which V_G is independent and consists exactly of the simplicial vertices of R. As in the first part, for all distinct vertices u and v in G,

$$uv \in E_G \Leftrightarrow d_H(u, v) \leq k - 1 \Leftrightarrow d_R(u, v) \leq k,$$

that is, R is a basic k-simplicial power of G. □

From Theorem 8 and its proof, we directly obtain:

Corollary 3. *Let $k \geq 2$ be an integer.*

(i) *A graph is a k-simplicial power of a block graph if and only if it is obtained from an induced subgraph of the $(k-1)$-th power of some block graph by replacing vertices by cliques.*
(ii) *A basic k-simplicial power of a block graph is the $(k-1)$-th power of a block graph if and only if it admits a basic k-simplicial block graph root in which each block contains exactly one simplicial vertex.*

5. 2-Simplicial powers of some subclasses of chordal graphs

By Theorem 7, every 3-leaf power is the 2-simplicial power of a claw-free block graph; Theorem 9 characterizes the more general class of 2-simplicial powers of block graphs as the (dart,gem)-free chordal graphs. Note that this graph class also appears under other names such as *strictly chordal graphs* in [24] and (4, 6)-*leaf powers* in [11]; it also has been characterized in terms of *contour vertices* in [13].

Theorem 9. *For every graph G, the following statements are equivalent:*

(i) *G is a 2-simplicial power of a block graph.*
(ii) *G is (dart,gem)-free chordal.*
(iii) *G arises from a block graph by replacing vertices by cliques.*

Proof. (i) ⇒ (ii): By Theorem 2 (i), squares of block graphs are chordal. We have to show that dart and gem are no 2-simplicial power of a block graph. It is easy to check that the diamond (see Fig. 4) with vertices a, b, c, d and edges ab, ac, ad, bc, bd has the unique 2-simplicial block root R as depicted in Fig. 4.

This shows that a diamond cannot be extended to a dart or a gem, hence (ii).

(ii) ⇒ (iii): The proof is by induction on the vertex number. Suppose that G is (dart, gem)-free chordal. If G is a block graph, we are done, hence let G contain an induced diamond (cf. Theorem 1), labelled as in Fig. 4 (left). If a, b are true twins in G, we are done by applying the induction hypothesis for $G - a$. Thus, we may assume that some vertex v is adjacent to a and non-adjacent to b. Note that, as G is chordal, v is non-adjacent to c or d. But then a, b, c, d and v induce a dart or a gem, a contradiction.

(iii) ⇔ (i): This is because of Corollary 3 (i) in Section 4. □

Theorem 10. *A graph is a 2-simplicial power of a ptolemaic graph if and only if it is ptolemaic.*

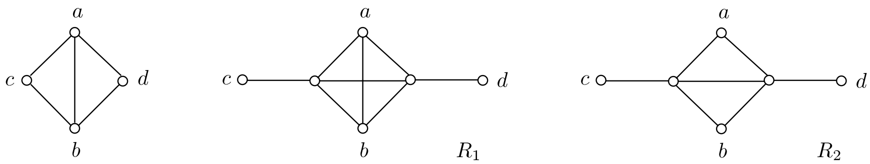

Fig. 5. The diamond and its two unique 2-simplicial ptolemaic roots.

Proof. *Necessity*. Suppose that G is a 2-simplicial power of a ptolemaic graph H. Then G is chordal (by Remark 1 and Theorem 2(i)). We have to show that the gem cannot be realized as a 2-simplicial power of a ptolemaic graph. It is easy to check that the diamond has exactly two 2-simplicial ptolemaic roots R_1, R_2 as depicted in Fig. 5.

This shows that a diamond cannot be extended to a gem, hence G is ptolemaic.

Sufficiency. Let $G = (V_G, E_G)$ be a ptolemaic graph. Let H be the graph obtained from the clique graph $K(G)$ and V_G by adding new edges vQ for those $v \in V_G$, $Q \in \mathcal{C}(G)$ with $v \in Q$. Note that V_G is a stable set in H. Moreover, it is easy to see that

$$V_G \text{ is the set of all simplicial vertices in } H, \tag{1}$$

and G is the 2-simplicial power of H: For all $u, v \in V_G$, $uv \in V_E$ if and only if $u, v \in Q$ for some $Q \in \mathcal{C}(G)$ if and only if $d_H(u, v) = 2$.

We now show that H is ptolemaic. First, by Theorem 3, the clique graph $K(G)$ of G is ptolemaic, hence H is chordal by (1). Next, we claim:

Claim: Let $\{Q_1, Q_2, Q_3\}$ be a triangle in $K(G)$ and let $u \in V_G$ such that $u \in (Q_1 \cap Q_2)$, $u \notin Q_3$. Then $Q_3 \cap Q_1 = Q_3 \cap Q_2$.

Proof of the claim: By the maximality of Q_3, there exists a vertex $q_3 \in Q_3 - (Q_1 \cup Q_2)$ nonadjacent to u. Since G is clique-Helly (cf. Theorem 2(ii)), there exists a vertex $v \in Q_1 \cap Q_2 \cap Q_3$. Assume, by contradiction, there exists a vertex $w \in (Q_3 \cap Q_1) - (Q_3 \cap Q_2)$. Then, by the maximality of Q_2, there exists $q_2 \in Q_2 - (Q_1 \cup Q_3)$ nonadjacent to w. Therefore, q_2, u, w, q_3 and v induce a gem (if $q_2q_3 \notin E_G$), or q_2, u, w, q_3 induce a 4-cycle in G (otherwise). This contradiction shows that $(Q_3 \cap Q_1) \subseteq (Q_3 \cap Q_2)$. By symmetry, $(Q_3 \cap Q_2) \subseteq (Q_3 \cap Q_1)$, and the claim is proved.

Finally, suppose to the contrary that H contains an induced gem with P_4 a, x, y, b with edges ax, xy, yb and universal vertex z. By (1) and the fact that $K(G)$ is ptolemaic, $x, y, z \in \mathcal{C}(G)$, and $a \in V_G$ or $b \in V_G$. Let $a \in V_G$, say. Then by the claim, $y \cap x = y \cap z$. Thus, if $b \in \mathcal{C}(G)$ then $b \cap y \cap z = b \cap y \cap x \neq \emptyset$, contradicting $b \cap x = \emptyset$. If $b \in V_G$ then, by the claim again, $b \in y \cap z = y \cap x$, contradicting $b \notin x$. Thus, H cannot contain an induced gem, and hence H is ptolemaic. □

Theorem 11. *For every graph G, the following statements are equivalent.*

(i) *G is a* 2*-simplicial power of a strongly chordal graph.*
(ii) *G is a* 2*-simplicial power of a strongly chordal split graph.*
(iii) *G is strongly chordal.*

Proof. (i) ⇒ (iii) and (ii) ⇒ (iii): This is because of Remark 1 and Theorem 2(i).

(ii) ⇒ (i): This implication is obvious.

(iii) ⇒ (ii): Let $G = (V_G, E_G)$ be a strongly chordal graph. Then $H = split(B_{\mathcal{C}}(G))$ is a strongly chordal split graph (cf. Theorem 2(iii)) whose simplicial vertices are exactly the vertices of G, and G is the 2-simplicial power of H (cf. proof of Proposition 1(i)). □

As chordal graphs, strongly chordal graphs, and ptolemaic graphs can be recognized in polynomial time, Theorems 9–11 imply

Corollary 4. 2*-simplicial powers of block graphs,* 2*-simplicial powers of ptolemaic graphs, and* 2*-simplicial powers of strongly chordal graphs can be recognized efficiently.*

Theorems 9–11 also give the following hierarchy; a *bull* is the graph on five vertices a, b, c, d, e and five edges ab, bc, cd, be, ce.

- 3-leaf powers (which are exactly the (bull,dart,gem)-free chordal graphs [6,18,29]) are a proper subclass of
- 2-simplicial powers of block graphs (which are exactly the (dart,gem)-free chordal graphs), and these are in turn a proper subclass of
- 2-simplicial powers of ptolemaic graphs (which are exactly the gem-free chordal graphs).

All these classes are proper subclass of strongly chordal graphs which coincide with the 2-simplicial powers of strongly chordal (split) graphs.

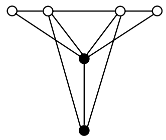

Fig. 6. The big bull and its black vertices.

6. Characterizing 3-simplicial powers of block graphs

This section deals with 3-simplicial powers of graphs which generalize the 4-leaf powers (cf. Theorem 7). The main result in section, Theorem 13, contains a characterization of 3-simplicial powers of block graphs that yields a polynomial-time recognition for such graphs; this characterization resembles, in a sense, a characterization of 4-leaf powers given in [10].

Observation 2. *Let G be a 3-simplicial power of a block graph, and let R be any 3-simplicial block graph root of G.*

(i) *For any two non-adjacent vertices x, y in G, all vertices in $N_G(x) \cap N_G(y)$ are adjacent to a common cut vertex in R.*
(ii) *Let x, y be the black vertices of an induced big bull in G; see Fig. 6. Then $d_R(x, y) \leq 2$.*

Proof. (i): Let $xv_1 \dots v_\ell y$ be the shortest path in R connecting x, y. Note that $\ell \geq 3$, and (i) follows from the following obvious facts: If $\ell \geq 5$, then $N_G(x) \cap N_G(y) = \emptyset$. If $\ell = 4$, then $N_G(x) \cap N_G(y)$ consists of exactly all simplicial vertices in R belonging to the block of R containing v_2v_3. If $\ell = 3$, then $N_G(x) \cap N_G(y)$ consists of exactly all simplicial vertices in R adjacent to v_2.

(ii): For vertex u in G write U for the unique block in R containing the simplicial vertex u. Let $abcd$ be the P_4 in the big bull, and let x be the black vertex of degree 4, y be the black vertex of degree 3 in the big bull. Assume to the contrary that $d_R(x, y) = 3$, and consider the shortest path xv_1v_2y in R connecting x and y. As b, c are common G-neighbors of x and y, $v_1 \in B$ or $v_2 \in B$ and $v_1 \in C$ or $v_2 \in C$. This and (i) (applied for a, y and for d, y) imply $v_1 \in B$, $v_1 \in C$. Thus, the blocks X, B, C have the cut vertex v_1 in common. Now, if $A \cap Z \neq \emptyset$ for some block Z containing v_1 (possibly $Z \in \{X, B, C\}$), then a is adjacent in G to all x, b, c. If no such block Z exists, a is adjacent in G to at most one of x, b, c. In any case, we get a contradiction, hence (ii). $\square$

Definition 3. A maximal clique Q in a graph $G = (V_G, E_G)$ is *special* if for all $x, y \in V_G - Q$ having a common neighbor in Q, $N(x) \cap Q = N(y) \cap Q$ or $|N(x) \cap Q| = 1$ or $|N(y) \cap Q| = 1$. A vertex v of G is *special* if $N[v]$ is a special clique in G.

Note that a special vertex is in particular simplicial. In Observation 2(i), the vertices in $N_G(x) \cap N_G(y)$ are not special (as they are not simplicial) and in (ii), the black vertex of degree 3 in the big bull is also not special. It turns out that special vertices play an important role in recognizing 2-connected basic 3-simplicial powers of block graphs. The following notation is useful in further discussions.

Notation. Let B be a block of a block graph R and x be a vertex in B. The connected component of $R - E_B$ containing x is denoted by $R_{B,x}$. The set of all simplicial vertices in R adjacent to x is denoted by $S_{B,x}$. Note that $S_{B,x}$ is a subset of $R_{B,x}$ and if $R_{B,x}$ is not a star at x, it has a simplicial vertex outside $S_{B,x}$.

In view of Corollary 3, we only need to consider basic 3-simplicial powers.

Observation 3. *Let $G = (V_G, E_G)$ be a 2-connected basic 3-simplicial power of a block graph and let R be a basic 3-simplicial block graph root of G. Let B be a block in R and x be a cut vertex of R in B. Then $S_{B,x} \neq \emptyset$. Moreover, if $R_{B,x}$ is not a star at x, then $|S_{B,x}| \geq 2$, and there is a simplicial vertex in $R_{B,x}$ outside $S_{B,x}$ adjacent, in G, to all vertices in $S_{B,x}$.*

Proof. Note that if $S_{B,x} = \emptyset$ or $R_{B,x}$ is not a star at x, then $R_{B,x} - S_{B,x} \neq \emptyset$. In any case, $S_{B,x}$ is a cutset of G separating $R_{B,x} - S_{B,x}$ and $V_G - R_{B,x}$. Hence, as G is 2-connected, $|S_{B,x}| \geq 2$, and some vertex v of G in $R_{B,x} - S_{B,x}$ must be adjacent to a vertex in $S_{B,x}$. It follows again by the 2-connectedness of G that there is such a vertex v adjacent to all vertices in $S_{B,x}$. $\square$

Lemma 2. *Let $G = (V_G, E_G)$ be a 2-connected basic 3-simplicial power of a block graph and let c be a vertex in G. Then G admits a basic 3-simplicial block graph root R such that $d_R(c, x) \geq 3$ for all $x \in V_G - c$ if and only if c is a special vertex of G.*

Moreover, if c is a special vertex of G, then there exists a basic 3-simplicial block graph root R of G such that c is a leaf of R and the cut vertex v in R adjacent to c belongs to exactly two blocks of R.

Proof. Assume that c is not special. If c is not simplicial, then, as G is 2-connected and chordal, c is a common neighbor of some non-adjacent vertices x and y in G with $|N_G(x) \cap N_G(y)| \geq 2$. If c is simplicial then c is the black vertex of degree 3 in some induced big bull in G. Thus, the if-part follows from Observation 2.

For the only-if part, let c be a special vertex in G. Consider a basic 3-simplicial block graph root R of G and let C be the block in R containing c. Let B be an arbitrary block of R such that C and B contain a common cut vertex, say v. Then

$$\text{if } B \text{ contains a cut vertex } x \neq v, \text{ then } R_{B,v} \text{ is a star at } v, \tag{2}$$

otherwise, $R_{B,v}$ has a cut vertex y adjacent to v. If $y \in C$, then by Observation 3, c has a G-neighbor in $R_{C,y}$. If $y \notin C$, let D be the block of $R_{B,v}$ containing v, y. By Observation 3 again, c has a G-neighbor in $R_{D,y}$. In any case, $N_G(c)$ is not complete because c has another G-neighbor in $R_{B,x}$, a contradiction.

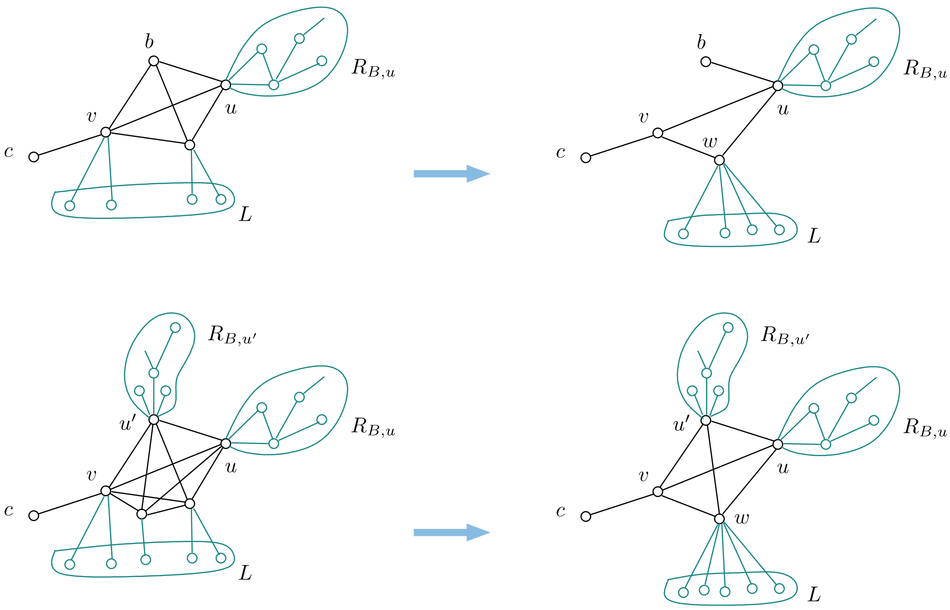

Fig. 7. Illustrating the last step in the proof of Lemma 2.

Furthermore,

if B contains a simplicial vertex, then for at most one cut vertex x of R in B, $R_{B,x}$ is not a star at x, (3)

otherwise, let b be the simplicial vertex of R in B and assume that there are cut-vertices $x_1 \neq x_2$ in B such that both R_{B,x_1} and R_{B,x_2} are not stars at x_1 and x_2, respectively. By (2), $x_1 \neq v$ and $x_2 \neq v$. By Observation 3, there exist simplicial vertices $c_i \in S_{B,x_i}$ and $c'_i \in R_{B,x_i} - S_{B,x_i}$ and c_i and c'_i are non-adjacent in G, $i = 1, 2$. But then $b, c_1, c_2 \in N_G(c)$ and $c'_1, c'_2 \notin N_G(c)$ give a contradiction to the assumption that c is a special vertex in G.

The proof of Lemma 2 is continued as follows. If every block B of R with $B \cap C \neq \emptyset$ has no cut-vertices in $B - C$, then G is a clique and we are done. Thus, there exists a block B with $B \cap C \neq \emptyset$ containing a cut vertex in $B - C$. Let v be the cut vertex in $B \cap C$. If for all cut-vertices x in B, $R_{B,x}$ is a star at x, then G is again a clique, and we are done.

So, let u be a cut vertex of R in B such that $R_{B,u}$ is not a star at u. By (2), $u \neq v$. Let L be the set of all leaves in $S_{B,v} - c$ and in $S_{B,x}$ for all cut-vertices $x \in B$ such that $R_{B,x}$ is a star at x.

Now, if B contains a simplicial vertex, say b, then, by (3), u is the unique cut vertex in B such that $R_{B,u}$ is not a star at u. In this case the block graph R' obtained from R by deleting the edge vb and the vertex set $(R_{B,v} - \{c, v\}) \cup \bigcup_{x \neq b,u,v} R_{B,x}$, and, if $L \neq \emptyset$, by adding a new vertex w and edges wv, wu and wz for all leaves $z \in L$ (see also Fig. 7) is clearly a desired basic 3-simplicial root for G.

If B contains no cut-vertices, the block graph R' obtained from R by deleting the vertex set $\bigcup_{R_{B,x} \text{is a } x\text{-star}} R_{B,x} - \{c, v\}$, and, if $L \neq \emptyset$, by adding a new vertex w and edges wv and wy for all y such that $R_{B,y}$ is not a star at y, and wz for all leaves $z \in L$ (see also Fig. 7) is clearly a desired basic 3-simplicial root for G. □

For a description of 2-connected basic 3-simplicial powers of block graphs, we need the following notion. A *split* of a graph $G = (V_G, E_G)$ is a partition of V_G into two disjoint sets V_1 and V_2 such that $|V_1| \geq 2$, $|V_2| \geq 2$ and the set of edges of G between V_1 and V_2 forms a complete bipartite graph. Graphs without split are called *prime*. A *simple split decomposition* of G by the split (V_1, V_2) is the decomposition of G into two graphs G_1 and G_2 where G_i is obtained from the subgraph of G induced by V_i and an additional vertex (a so-called marker) v by adding all edges between v and those vertices in V_i which have a neighbor in $G - V_i$. Split decomposition can be computed in linear time [16].

We characterize 2-connected basic 3-simplicial powers of block graphs by reducing to smaller 2-connected basic 3-simplicial powers of block graphs as follows.

Theorem 12. *A 2-connected graph $G = (V_G, E_G)$ is a basic 3-simplicial power of a block graph if and only if*

(i) *G is the square of a block graph, or*
(ii) *G has a special vertex v such that $N_G(v) = N_G(x) \cap N_G(y)$ for some non-adjacent vertices x and y, and $G - v$ is a 2-connected basic 3-simplicial power of a block graph, or*
(iii) *G admits a split (V_1, V_2) such that G_1 and G_2 are 2-connected basic 3-simplicial powers of block graphs and the marked vertex is special in both G_1 and G_2.*

Proof. *Necessity*. Assume that condition (i) does not hold (in particular, G is not complete). Then, by Corollary 3(ii), any basic 3-simplicial block graph root of G has a block that contains no simplicial vertices.

Let R be a basic 3-simplicial block graph root of G with minimum number of blocks that contain no simplicial vertices, and consider such a block B in R. Let $V_B = \{b_1, \ldots, b_q\}$, $q \geq 2$. Set $S_i = S_{B,b_i}$ and $R_i = R_{B,b_i}$, $1 \leq i \leq q$. Note that S_i are pairwise non-empty disjoint sets because B contains no simplicial vertex and G is 2-connected, and it is easy to see that

$$Q = G[S_1 \cup \cdots \cup S_q] \text{ is a special clique.} \tag{4}$$

Moreover, at least one of S_i consists of at least two simplicial vertices: If not, as G is not complete, R has a simplicial vertex v not adjacent to all b_i. Let v belong to R_1, say. Then, in G, S_1 separates v from S_2, contradicting the 2-connectedness of G.

Thus, let $|S_1| \geq 2$, say. Let V_1 be the set of all simplicial vertices in R_1, $V_2 := V_G - V_1$. Now, if $|V_2| \geq 2$, then (V_1, V_2) is a split of G (as the edges of G between V_1 and V_2 are exactly the edges between S_1 and $S_2 \cup \cdots \cup V_q$). Note that the corresponding marked graphs G_i, $i = 1, 2$, are 2-connected (as G is 2-connected) and basic 3-simplicial powers of block graphs (as they are isomorphic to an induced subgraph in G). Moreover, by (4), the marked vertex in G_i is special, hence (iii) holds.

Consider the case $|V_2| = 1$. Then $q = 2$ and $|S_2| = 1$, say $S_2 = \{v\}$. Hence R_2 consists of exactly v and b_2. Then, no simplicial vertex in S_1 is a leaf of R: If $a \in S_1$ is a leaf, then the block graph $R' = R + b_2a$ is also a basic 3-simplicial root of G with less number of blocks containing no simplicial vertices than R, a contradiction. Fix two simplicial vertices $a \neq b$ in S_1, and let A and B be the blocks in R_1 containing a, b_1 and b, b_1, respectively. Let $u \in A - b_1$ be a cut vertex of R_1. If all blocks containing u contain no simplicial vertices, then, in G, a separates v from the simplicial vertices in $R_{A,u}$, a contradiction. Thus, there is a simplicial vertex $x \neq a$ in R_1 adjacent to u. Similarly, there is a simplicial vertex $y \neq b$ in R_1 adjacent to a cut vertex in $B - b_1$. Clearly, $N_G(v) = N_G(x) \cap N_G(y)$, and (ii) holds.

Sufficiency. If (i) is satisfied, G is a basic 3-simplicial power of a block graph by Theorem 8.

If (ii) is satisfied, let R be a basic 3-simplicial block graph root of $G - v$. By Observation 2, all vertices in $N_G(v) = N_G(x) \cap N_G(y)$ are adjacent in R to a common cut vertex z in R. Moreover, any simplicial vertex in R adjacent to z must be adjacent in G to x and y, hence such a simplicial vertex must be a G-neighbor of v, too. Then the block graph obtained from R by adding v and a new vertex v' and edges vv', $v'z$ is a basic 3-simplicial root for G.

If (iii) is satisfied, let v be the marked vertex in G_i, $i = 1, 2$. By Lemma 2, G_i admits a basic 3-simplicial block graph root R_i such that v is a leaf in R_i and the cut vertex v_i in R_i adjacent to v belongs to exactly two blocks in R_i. Let B_i be the block in R_i containing v_i but not v. Then the block graph R obtained from R_1 and R_2 by deleting v, v_1, v_2 and joining all edges between $B_1 - v_1$ and $B_2 - v_2$ is clearly a basic 3-simplicial root for G. □

Theorem 12 gives a recursive procedure that tests in time $\mathcal{O}(n^3)$ if a 2-connected chordal graph G with n vertices is the basic 3-simplicial power of a block graph: Testing if G is the square of a block graph can be done in linear time by a result in [25]. If G is not the square of a block graph, check if G satisfies (ii) or (iii). If yes, recursively test the corresponding 2-connected graphs $G - v$, and G_1 and G_2, respectively. Whether a maximal clique is special can be easily tested in time $\mathcal{O}(n^2)$, the at most n maximal cliques in a chordal graph can be found in linear time, and testing (ii) and (iii) can be done in time $\mathcal{O}(n^3)$.

Theorem 13 contains a characterization of basic 3-simplicial powers of block graphs via reducing to the 2-connected case. We need a further property of 2-connected basic 3-simplicial powers of block graphs, that can be derived from the proof of Lemma 2.

Observation 4. *Every 2-connected basic 3-simplicial power of a block graph $G = (V_G, E_G)$ admits a basic 3-simplicial block graph root R such that, for all special vertices c of G and all $x \in V_G - c$, $d_R(c, x) \geq 3$.*

Theorem 13. *For every graph G, the following statements are equivalent.*

(i) *G is a basic 3-simplicial power of a block graph.*
(ii) *G is an induced subgraph of the square of a block graph.*
(iii) *Each block of G is a basic 3-simplicial power of a block graph, and each cut vertex v of G is non-special in at most one block containing v.*

Proof. The equivalence (i) ⇔ (ii) is shown by Theorem 8.

(i) ⇒ (iii): We have only to show the cut vertex condition. Let v be a cut vertex of G, and consider two blocks $A \neq B$ in G containing v. Consider a basic 3-simplicial block graph root R of G, and let R_A and R_B be the smallest induced subgraphs of R containing V_A and V_B, respectively. Then R_A and R_B are basic 3-simplicial block graph roots of of A and B respectively. Now, if v is non-special in both A and B, then by Lemma 2, $d_{R_A}(v, a) = 2$ and $d_{R_B}(v, b) = 2$ for some vertices $a \in A - v$ and $b \in B - v$. But, as v is simplicial in R, it follows $d_R(a, b) \leq 3$, a contradiction.

(iii) ⇒ (i): For each block B of G let R_B be a basic 3-simplicial block graph root of B such that $d_{R_B}(a, x) \geq 3$ for all special vertices $a \in B$ and all $x \in B - a$ (cf. Observation 4).

A basic 3-simplicial block graph root R for G is now obtained by putting all R_B together as follows. For each cut vertex v of G, let $B_1, \ldots, B_q$ be the blocks of G containing v. Identify the vertex v from the roots R_{B_i} and making all blocks in R_{B_i} containing v to a clique. Let the resulting graph be R. As the block-cut-vertex graph of a graph is a tree (cf. [22]), R is a block graph.

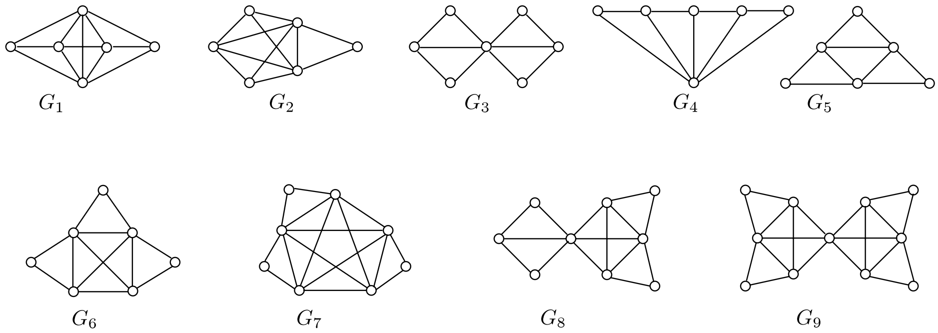

Fig. 8. Forbidden subgraphs $G_1, \ldots, G_9$ characterize induced subgraphs of squares of block graphs.

We claim that R is a 3-simplicial block graph root for G. It is clear from the construction of R that the adjacencies within a block of G are preserved by R. Consider vertices $x \in B - B'$ and $y \in B' - B$, where B and B' are two distinct blocks of G containing a same cut vertex v. As v is special in at least one of B, B', $d_{R_B}(v, x) \geq 3$ or $d_{R_{B'}}(v, y) \geq 3$. Hence, from the construction of R, $d_R(x, y) = d_{R_B}(v, x) + d_{R_{B'}}(v, y) \geq 5 - 2 + 1 = 4$. □

As the 2-connected components of a given graph can be computed in linear time, Theorems 12, 13 and Corollary 3(i) together imply:

Corollary 5. 3-*simplicial powers of block graphs can be recognized efficiently.*

Remark 3. In the full version of [11], induced subgraphs of squares of block graphs (which appear in Theorem 13) are also characterized in terms of forbidden subgraphs as follows:

A graph is an induced subgraph of the square of some block graph if and only if it is $(G_1, G_2, \ldots, G_9)$*-free chordal; see* Fig. 8.

7. Conclusion and open problems

The notion of simplicial powers of graphs is introduced for the first time in this paper, motivated by the study of leaf powers. It turns out that simplicial powers of block graphs are a natural generalization of leaf powers; cf. Theorem 7.

Since both leaf powers and simplicial powers of block graphs are strongly chordal, and as indicated by the descriptions of 2- and 3-simplicial powers of block graphs (Theorems 9 and 13), simplicial powers of block graphs are 'not too far' from leaf powers. Thus, our results on simplicial powers of block graphs might shed new light on the open problem of characterizing k-leaf powers for $k \geq 5$ and the open problem of recognizing k-leaf powers for $k \geq 6$.

Besides the obvious open problems of characterization and recognition for k-simplicial powers of block graphs, $k \geq 4$, another open question is as follows. As mentioned in the introduction, not every strongly chordal graph is a leaf power. Is every strongly chordal graph a k-simplicial power of a block graph, for some k? An answer to this question will also indicate 'how far' indeed are simplicial powers of block graphs from leaf powers.

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